Some modeling for second order differential equations

On the occasion of my daughter's(urologist) wedding,
I wish a Happy Marriage to **Koichi**(cardiologist) and **Erina**.

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ABSTRACT

Simplistic differential equation models are often used to introduce concepts and principles which are important for understanding the dynamic behavior of complex physical systems. In this short note, we employ such a model to study the response of a tall building due to horizontal seismic at the foundation generated by an earthquake due to the book of "Differential Equations by Brannan and Boyce".

1.Model (assumption)

Figure 1 is an illustration of a building idealized as a collection of n floors, each of mass m, connected together by vertical walls. If we neglect gravitation and restrict motion to the horizontal direction, the displacements of the floors, relative to a fixed frame of reference, zero and the floors are in perfect vertical alignment.

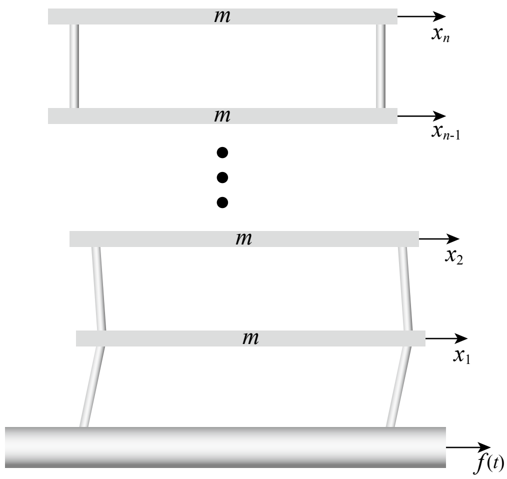

Figure A building consisting of floors of mass m connected by stiff but flexible vertical walls.

Key words and phrases. Matrix differential equations, undamped building, earthquake.

When adjacent floor are not in alignment, we assume that the walls exert a flexural restoring force proportional to the difference in displacements between the floors with proportionality constant k. Thus the equation of motion for the j th floor is

$$mx_j'' = -k(x_j - x_{j-1}) - k(x_j - x_{j+1}),\ \ j = 2, \ldots, n-1 \tag{1}$$

which the equations for the first floor and nth, or top, floor

$$mx_1'' = -k(x_1 - f(t)) - k(x_1 - x_2)\,, \tag{2}$$

and

$$mx_n'' = -k(x_n - x_{n-1})\,, \tag{3}$$

respectively. The horizontal motion of the foundation generated by the earthquake is described by the input function $f(t)$.

The Undamped Building.

Project(A)

To show that Equations (1) through (3) can be expressed in matrix notation as

$$\boldsymbol{x}'' + \omega_0^2 \mathbf{K}\boldsymbol{x} = \omega_0^2 f(t)\mathbf{z} \quad \text{(i)}$$

where

$$\omega_0^2 = \frac{k}{m}\,,\ \ \boldsymbol{x} = (x_1,\ x_2, \ldots, x_n)^{\mathrm{T}},\ \ \mathbf{z} = (1, 0, \ldots, 0)^{\mathrm{T}},$$

and

$$\mathbf{K} = \begin{pmatrix} 2 & -1 & 0 & 0 & . & . & . & 0 \\ -1 & 2 & -1 & 0 & . & . & . & 0 \\ 0 & -1 & 2 & -1 & . & . & . & 0 \\ . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & -1 & 2 & -1 \\ 0 & . & . & . & . & 0 & -1 & 1 \end{pmatrix}. \quad \text{(ii)}$$

Consideration:

$$\boldsymbol{x} = col(x_1, x_2, \ldots, x_n) = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \in \mathbf{R}^n$$

where

$$x_1 = x_1(t),\ x_2 = x_2(t), \ldots,\ x_n = x_n(t)$$

Let $j=1$,

$$mx_1'' = -k(x_1 - f(t)) - k(x_1 - x_2)$$

$$\therefore\ x_1'' = -\frac{2k}{m}x_1 + \frac{k}{m}x_2 + \frac{k}{m}f(t)$$

$j=2$,

$$mx_2'' = -k(x_2 - x_1) - k(x_2 - x_3)$$

$$\therefore\ x_2'' = -\frac{2k}{m}x_2 + \frac{k}{m}x_1 + \frac{k}{m}x_3$$

$j=3$,

$$mx_3'' = -k(x_3 - x_2) - k(x_3 - x_4)$$

$$\therefore\ x_3'' = -\frac{2k}{m}x_3 + \frac{k}{m}x_2 + \frac{k}{m}x_4$$

$$\cdot$$

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$j=n-1$,

$$mx_{n-1}'' = -k(x_{n-1} - x_{n-2}) - k(x_{n-1} - x_n)$$

$$\therefore\ x_{n-1}'' = -\frac{2k}{m}x_{n-1} + \frac{k}{m}x_{n-2} + \frac{k}{m}x_n$$

$j=n$,

$$mx_n''=-k(x_n-x_{n-1})$$

$$\therefore \quad x_n''=-\frac{k}{m}x_n+\frac{k}{m}x_{n-1}$$

Consequently, we have the following matrix expression.

$$\boldsymbol{x}''(t)=\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1}'' \\ x_n \end{pmatrix}=\begin{pmatrix} -\frac{2k}{m} & \frac{k}{m} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \frac{k}{m} & -\frac{2k}{m} & \frac{k}{m} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \frac{k}{m} & -\frac{2k}{m} & \frac{k}{m} & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{k}{m} & -\frac{2k}{m} & \frac{k}{m} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{k}{m} & -\frac{k}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{pmatrix}+\begin{pmatrix} \frac{k}{m}f(t) \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix},$$

$$\boldsymbol{x}''(t)+\begin{pmatrix} \frac{2k}{m} & -\frac{k}{m} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{k}{m} & \frac{k}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{pmatrix}=\begin{pmatrix} \frac{k}{m}f(t) \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix},$$

Let $\omega_0^2=\dfrac{k}{m}$, $\mathbf{z}=col(1, 0, 0, \dots, 0)^{\mathrm{T}}=\begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$, Then we have

$$
\boldsymbol{x}''(t)+\begin{pmatrix}
2\omega_0^2-\omega_0^2 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
-\omega_0^2 & 2\omega_0^2-\omega_0^2 & 0 & 0 & \cdot & \cdot & 0 \\
0 & -\omega_0^2 & 2\omega_0^2-\omega_0^2 & 0 & 0 & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & 0 & 0 & -\omega_0^2 & 2\omega_0^2-\omega_0^2 \\
0 & 0 & 0 & 0 & 0 & 0 & -\omega_0^2 \quad \omega_0^2
\end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{pmatrix}=\begin{pmatrix} \frac{k}{m}f(t) \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix},
$$

$$
\boldsymbol{x}''(t)+\omega_0^2\begin{pmatrix}
2 & -1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
-1 & 2 & -1 & 0 & 0 & \cdot & \cdot & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{pmatrix}=\omega_0^2 f(t)\begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix},
$$

Finally, we have $\boldsymbol{x}''(t)+\omega_0^2\mathbf{K}\boldsymbol{x}(t)=\omega_0^2 f(t)\mathbf{z}$ □

Project(B)

Show that K in (ii) satisfies $\boldsymbol{x}^{\mathrm{T}}\mathbf{K}\boldsymbol{x}=\sum_{i=1}^{n-1}\{(x_i-x_{i+1})^2+{x_1}^2\}$ and is therefore positive definite.

Consideration

To show this, just direct calculation.

Namely,

$$\boldsymbol{x} = (x_1,\ x_2,\ \ldots\ x_{n\text{-}1}\ x_n)^{\mathrm{T}} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{pmatrix}$$

$$\mathbf{K}x = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 - x_4 \\ -x_3 + 2x_4 - x_5 \\ \cdot \\ \cdot \\ \cdot \\ -x_{n-2} + 2x_{n-1} - x_n \\ -x_{n-1} + x_n \end{pmatrix}.$$

Then,

$$x^{\mathrm{T}}\mathbf{K}x = (x_1,\ x_2,\ x_3, \ldots, x_{n-1},\ x_n) \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 - x_4 \\ -x_3 + 2x_4 - x_5 \\ \cdot \\ \cdot \\ \cdot \\ -x_{n-2} + 2x_{n-1} - x_n \\ -x_{n-1} + x_n \end{pmatrix}$$

$$= 2(x_1{}^2 + x_2{}^2 + \ldots + x_{n-1}{}^2) + x_n{}^2 - (2x_1x_2 + 2x_2x_3 + \ldots + 2x_{n-1}x_n)$$

$$= x_1{}^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \ldots + (x_{n-1} - x_n)^2$$

$$= \sum_{i=1}^{n-1} (x_i - x_{i+1})^2 + x_1{}^2 > 0$$

for every real n-vector $\boldsymbol{x} \neq \mathbf{0}$.

Project(C)

Show that if K is positive definite and λ and $\boldsymbol{u}$ are an eigenvalue-eigenvector pair for K,

then $\lambda = \dfrac{\boldsymbol{u}^{\mathrm{T}}\mathbf{K}\boldsymbol{u}}{\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}} > 0$.

Consideration

Let's take inner product of $\mathbf{K}\boldsymbol{u}$ with $\boldsymbol{u}$,

$$\therefore \ (\mathbf{K}\boldsymbol{u}, \boldsymbol{u}) = (\lambda\boldsymbol{u}, \boldsymbol{u})$$

$$\therefore \ (\mathbf{K}\boldsymbol{u}, \boldsymbol{u}) = \lambda(\boldsymbol{u}, \boldsymbol{u})$$

So, if $\boldsymbol{u} \neq \mathbf{0}$, that is $(\boldsymbol{u}, \boldsymbol{u}) \neq 0$

$$\lambda = \frac{(\mathbf{K}\boldsymbol{u}, \boldsymbol{u})}{(\boldsymbol{u}, \boldsymbol{u})} = \frac{\boldsymbol{u}^{\mathrm{T}}\mathbf{K}\boldsymbol{u}}{\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}}$$

and fact $\mathrm{K} > 0$ implies $\boldsymbol{u}^{\mathrm{T}}\mathbf{K}\boldsymbol{u} > 0$, but also $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} > 0$ $\quad \therefore \lambda > 0$

Project(D)

For the case $n = 3$, demonstrate numerically that the eigenvalues of $\mathbf{K}$, $\lambda_j = \omega_j^2$, $j = 1, 2, 3$ can be ordered as follows, $0 < \omega_1^2 < \omega_2^2 < \omega_3^2$.

Consideration

$n = 3$, we have $\mathbf{K}$ as follows as

$$\mathbf{K} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Calculate eigenvalue of $\mathbf{K}$

$$\lambda = \det(\lambda\mathbf{I} - \mathbf{K})$$

$$= \det\begin{vmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \end{vmatrix} = 0\,.$$

Then we have $\lambda^3 - 5\lambda^2 + 6\lambda - 1 = 0$.

There are several methods that how to solve the above 3^{rd-}order equation.

We can confirm that there are three distinct positive real number solutions.

So, we can have the result. That is $0 < \omega_1^2 < \omega_2^2 < \omega_3^2$.

<Notice> In general case, we have to show that

$0 < \omega_1^2 < \omega_2^2 < \omega_3^2 < \ldots < \omega_{n-1}^2 < \omega_n^2$ by “graphing the polynomial.

Project(E)

$\mathbf{K}$ is real and symmetric matrix, then $\mathbf{K}$ possesses a set of n orthogonal eigenvectors $[\,\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \ldots, \boldsymbol{u}_{n-1}, \boldsymbol{u}_n\,]$.

These eigenvectors can make vector space, we call “**normal mode representation**”,

$$\boldsymbol{x}(t) = a_1(t)\boldsymbol{u}_1 + a_2(t)\boldsymbol{u}_2 + \ldots + a_n(t)\boldsymbol{u}_n \quad (*)$$

Also, (*) is the solution of Differential Equation

$$\begin{aligned} &\boldsymbol{x}'' + \omega_0^2\mathbf{K}\boldsymbol{x} = \omega_0^2 f(t)\mathbf{z} \\ &\boldsymbol{x}(0) = \boldsymbol{x}_0 \quad \boldsymbol{x}'(0) = \boldsymbol{v}_0 \end{aligned} \qquad \text{(i)}$$

Replace $\boldsymbol{x}$ by $\sum_{i=1}^{n} a_i(t)\boldsymbol{u}_i$ in (i)

$$\left(\sum_{i=1}^{n} a_i(t)\boldsymbol{u_i}\right)'' + \omega_0^2\,\mathbf{K}\left(\sum_{i=1}^{n} a_i(t)\boldsymbol{u_i}\right) = {\omega_0}^2 f(t)\mathbf{z}$$

$$\sum_{i=1}^{n} a_i''(t)\boldsymbol{u_i} + \omega_0^2 \sum_{i=1}^{n} a_i(t)\,\mathbf{K}\boldsymbol{u_i} = {\omega_0}^2 f(t)\mathbf{z}$$

But $\mathbf{K}\boldsymbol{u_i}=\lambda_i\boldsymbol{u_i}$, where $\lambda_i=\omega_i^2$ is i^{th} eigenvalue of $\mathbf{K}$

Then we have

$$\sum_{i=1}^{n} a_i''(t)\,\boldsymbol{u_i} + \omega_0^2 \sum_{i=1}^{n} a_i(t)\lambda_i\boldsymbol{u_i} = \omega_0^2 f(t)\mathbf{z}$$

$$\sum_{i=1}^{n} a_i''(t)\,\boldsymbol{u_i} + \omega_0^2 \sum_{i=1}^{n} a_i(t)\omega_i^2\,\boldsymbol{u_i} = \omega_0^2 f(t)\mathbf{z}$$

$$\sum_{i=1}^{n} (\,a_i''(t) + \omega_0^2\,a_i(t)\omega_i^2\,)\boldsymbol{u_i} = \omega_0^2 f(t)\mathbf{z} \quad (*)$$

And break down into components means

$$(\,a_j''(t) + \omega_0^2\,a_j(t)\omega_j^2\,)u_j = \omega_0^2 f(t)\mathrm{z}_j$$

Take inner product of both sides of (*) with $\boldsymbol{u_j}$

Then we have,

$$\sum_{i=1}^{n}(a_i''(t) + \omega_0^2\;a_i(t)\omega_i^2)(\boldsymbol{u_i}\,,\boldsymbol{u_j}) = \omega_0^2 f(t)(\boldsymbol{z}\,,\boldsymbol{u_j}) \;\; for\;\; 0 \leqq i,\; j \leqq n$$

But $\boldsymbol{u}_i$ are orthogonal, that is $(\boldsymbol{u_i}\,,\boldsymbol{u_j})=0$ when $i \neq j$

So, we simplify

$$(a_j''(t) + \omega_0^2\;a_j(t)\omega_j^2)(\boldsymbol{u_j}\,,\boldsymbol{u_j}) = \omega_0^2 f(t)(\mathbf{z}\,,\boldsymbol{u_j})$$

$$(a_j''(t) + \omega_0^2 a_j(t)\omega_j^2) = \omega_0^2 f(t)\frac{(\mathbf{z}\,,\boldsymbol{u_j})}{(\boldsymbol{u_j}\,,\boldsymbol{u_j})},$$

where $z_j = \dfrac{(\mathbf{z}\,,\boldsymbol{u_j})}{(\boldsymbol{u_j}\,,\boldsymbol{u_j})} = \dfrac{\boldsymbol{u_j}^{\mathrm{T}}\!\bullet \mathbf{z}}{\boldsymbol{u_j}^{\mathrm{T}}\!\bullet \boldsymbol{u_j}}$

Let’s take the mode amplitude $a_i(t)$.

Given initial conditions $\boldsymbol{x(0)} = \boldsymbol{x_0}$ implies $\sum_{i=1}^{n} a_i(0)\boldsymbol{u_i} = \boldsymbol{x_0}$

As before take inner product of both sides with respect to $\boldsymbol{u_j}$, we have

$$\sum_{i=1}^{n} a_i(0)(\boldsymbol{u_i}, \boldsymbol{u_j}) = (\boldsymbol{x_0}, \boldsymbol{u_j}) \qquad (**)$$

Then

$$a_1(0)(\boldsymbol{u}_1, \boldsymbol{u}_j) + a_2(0)(\boldsymbol{u}_2, \boldsymbol{u}_j) + \ldots + a_n(0)(\boldsymbol{u}_n, \boldsymbol{u}_j)$$

$$= a_j(0)(\boldsymbol{u}_j, \boldsymbol{u}_j) = (\boldsymbol{u}_0, \boldsymbol{u}_j)$$

$$\therefore \boldsymbol{a}_j(0) = \frac{(\boldsymbol{x}_0, \boldsymbol{u}_j)}{(\boldsymbol{u}_j, \boldsymbol{u}_j)} = \frac{(\boldsymbol{u}_j^{\mathrm{T}} \cdot \boldsymbol{x}_0)}{(\boldsymbol{u}_j^{\mathrm{T}} \cdot \boldsymbol{u}_j)} = \alpha_j .$$

Similarly we do derivative for (**)

$\boldsymbol{x}'(0) = \boldsymbol{v}_0$ implies that

$$\sum_{i=1}^{n} \boldsymbol{a}_i'(0)\boldsymbol{u}_i = \boldsymbol{v}_0$$

$$\sum_{i=1}^{n} \boldsymbol{a}_i'(0)(\boldsymbol{u}_i, \boldsymbol{u}_j) = (\boldsymbol{v}_0, \boldsymbol{u}_j)$$

Then,

$$\boldsymbol{a}_1'(0)(u_1, u_j) + \boldsymbol{a}_2'(0)(u_2, u_j) + \ldots + \boldsymbol{a}_n'(0)(u_n, u_j)$$

$$\boldsymbol{a}_j'(0)(\boldsymbol{u}_i, \boldsymbol{u}_j) = (\boldsymbol{v}_0, \boldsymbol{u}_j)$$

$$\boldsymbol{a}_j'(0) = \frac{(\boldsymbol{v}_0, \boldsymbol{u}_j)}{(\boldsymbol{u}_j, \boldsymbol{u}_j)} = \frac{\boldsymbol{u}_j^{\mathrm{T}} \cdot \boldsymbol{v}_0}{\boldsymbol{u}_j^{\mathrm{T}} \cdot \boldsymbol{u}_j} = \beta_j \qquad \square$$

Conclusion

This time I only considered when in the case of the undamped building. As we saw, the motion of earthquake to undamped building can be expressed the 2nd order differential equations.

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