On Legendre's formula and distribution of prime numbers

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Abstract

The conclusion in this paper is based on the several idea obtained in recherché with respect to the prime number. This paper consists of four topics. First we will consider Legendre's formula of prime numbers, and then we will arrive at a certain inequality of prime counting function $\pi(x)$. We will see relevance $\pi(x)$ and $\frac{x}{\log x}$ there. In the third section we will consider Legendre's conjecture, finally we will give a new representation of $\pi(x)$ with Fourier series expansion.

1. Legendre's formula of prime numbers

Prime counting function $\pi(x)$ gives number of primes less than given any real number x. For example, $\pi(10) = 4$ because there exists 4 primes in the case of 10 $(p = 2, 3, 5, 7)$. Let N be a natural number. Any composite number less than N is able to be divided by prime less than $\sqrt{N}$. This property gives recurrence formula of $\pi(x)$ as follows. This formula was found out by Legendre.

1: He was a student at The High School of CHUO University from 1992 to 1995.

Legendre's Formula

$$\pi(N) = \pi(\sqrt{N}) - 1 + \sum_{\substack{q=\prod p_i \\ p_i \leq \sqrt{N}}} \mu(q) \left[\frac{N}{q}\right] \tag{1.1}$$

where μ is a Möbius function, q is 1 or product of several primes less than $\sqrt{N}$,

and $\left[\frac{N}{q}\right]$ is a maximum integer that is not exceeded $\frac{N}{q}$.

The concept of Legendre's formula is based on Eratosthenes' sieve and above property to be divisible by prime less than $\sqrt{N}$.

Let's see an example at first.

Actually in the case of $N = 100$, number of primes less than 100 is $25(p = 2, 3, 5, 7 \cdots 97)$ $i.e.$ $\pi(100) = 25$.

Next, we will calculate right side of (1.1) under this condition,

The first term of the right side of (1.1) is equal to numbers of the primes less than 10 $(=\sqrt{100})$, so it is 4 $(p = 2, 3, 5, 7)$. q is product of some of primes less than 10 like the following table.

Number of primes combined	Value of q
0	1
1	2, 3, 5, 7
2	6, 10, 14, 15, 21, 35
3	30, 42, 70, 105
4	210

Thus

$$\sum \mu(q)\left[\frac{N}{q}\right]$$

$$= \mu(1)\left[\frac{100}{1}\right]$$

$$+\mu(2)\left[\frac{100}{2}\right] + \mu(3)\left[\frac{100}{3}\right] + \mu(5)\left[\frac{100}{5}\right] + \mu(7)\left[\frac{100}{7}\right]$$

$$+\mu(6)\left[\frac{100}{6}\right] + \mu(10)\left[\frac{100}{10}\right] + \mu(14)\left[\frac{100}{14}\right] + \mu(15)\left[\frac{100}{15}\right] + \mu(21)\left[\frac{100}{21}\right] + \mu(35)\left[\frac{100}{35}\right]$$

$$+\mu(30)\left[\frac{100}{30}\right] + \mu(42)\left[\frac{100}{42}\right] + \mu(70)\left[\frac{100}{70}\right] + \mu(105)\left[\frac{100}{105}\right]$$

$$+\mu(210)\left[\frac{100}{210}\right]$$

$$= 1 \cdot 100$$

$$+(-1) \cdot 50 + (-1) \cdot 33 + (-1) \cdot 20 + (-1) \cdot 14$$

$$+1 \cdot 16 + 1 \cdot 10 + 1 \cdot 7 + 1 \cdot 6 + 1 \cdot 4 + 1 \cdot 2$$

$$+(-1) \cdot 3 + (-1) \cdot 2 + (-1) \cdot 1 + (-1) \cdot 0$$

$$+1 \cdot 0$$

$$= 22$$

Third term is 22. Therefore the right side of (1.1) gives 25.

2. Inequality of $\pi(N)$ with logarithmic function

The purpose of this chapter is to prove a certain inequality with respect to the relevance of $\pi(x)$ and $\frac{x}{\log x}$ by using Legendre's formula.

Now, if we focus on the third term of Legendre's formula (1.1), we see relevance of between $\left[\frac{N}{q}\right]$ and $\frac{N}{q}$ as follows,

$$\left[\frac{N}{q}\right] \leq \frac{N}{q} < \left[\frac{N}{q}\right] + 1 \tag{2.1}$$

Subtract $\left[\frac{N}{q}\right] + \frac{1}{2}$ from both sides of (2.1), then we get

$$-\frac{1}{2} \leq \left(\frac{N}{q} - \frac{1}{2}\right) - \left[\frac{N}{q}\right] < \frac{1}{2} \tag{2.2}$$

Next we define $\pi^*(N)$ as follows,

Definition

$$\pi^*(N) = \pi(\sqrt{N}) - 1 + \sum_{\substack{q=\prod p_i \\ p_i \leq \sqrt{N}}} \mu(q)\left(\frac{N}{q} - \frac{1}{2}\right) \tag{2.3}$$

Let's prove the next lemma with respect to the third term of (2.3), before we discuss about relevance $\pi(N)$ and $\pi^*(N)$.

Lemma2.1

$$-2^{\pi(\sqrt{N})-1} < \sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q)\left(\frac{N}{q}-\frac{1}{2}\right) - \sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q)\left[\frac{N}{q}\right] < 2^{\pi(\sqrt{N})-1} \tag{2.4}$$

Proof

$$\sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q)\left(\frac{N}{q}-\frac{1}{2}\right) - \sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q)\left[\frac{N}{q}\right] = \sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q)\left\{\left(\frac{N}{q}-\frac{1}{2}\right) - \left[\frac{N}{q}\right]\right\} \tag{2.5}$$

We will estimate $\mu(q)\left\{\left(\frac{N}{q}-\frac{1}{2}\right) - \left[\frac{N}{q}\right]\right\}$ of right side of (2.5). By using (2.2) and we notice sign of $\mu(q)$,

If $\mu(q) = 1$,

$$-\frac{1}{2} \le \mu(q)\left\{\left(\frac{N}{q}-\frac{1}{2}\right) - \left[\frac{N}{q}\right]\right\} < \frac{1}{2}$$

If $\mu(q) = -1$,

$$-\frac{1}{2} < \mu(q)\left\{\left(\frac{N}{q}-\frac{1}{2}\right) - \left[\frac{N}{q}\right]\right\} \le \frac{1}{2}$$

Thus we can get inequality as follows by taking sum for all q,

$$-\frac{1}{2}\cdot 2^{\pi(\sqrt{N})} < \sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q)\left\{\left(\frac{N}{q}-\frac{1}{2}\right) - \left[\frac{N}{q}\right]\right\} < \frac{1}{2}\cdot 2^{\pi(\sqrt{N})}$$

We obtain (2.4) because both sides of above inequality is equal to $2^{\pi(\sqrt{N})-1}$.

□

Therefore we see following proposition with respect to $\pi(N)$ and $\pi^*(N)$.

Proposition2.2

$$-2^{\pi(\sqrt{N})-1} < \pi^*(N) - \pi(N) < 2^{\pi(\sqrt{N})-1} \tag{2.6}$$

Next, we will prove following theorem with respect to $\pi(N)$.

Theorem2.3

For any $\varepsilon > 0$ *, there exists a natural number* N_0 *such that* $N \geq N_0$ *for all* N *satisfying the following inequality.*

$$\pi(\sqrt{N}) - 1 + N \cdot \left(\frac{C}{logN} - \varepsilon\right) - 2^{\pi(\sqrt{N})-1} < \pi(N) < \pi(\sqrt{N}) - 1 + N \cdot \left(\frac{C}{logN} + \varepsilon\right) + 2^{\pi(\sqrt{N})-1} \tag{2.7}$$

where $C = 2e^{-\gamma}$ (γ *is Euler's constant)*

Before we will prove Theorem2.3, we need two lemmas.

Lemma2.4 (Mertens' Theorem)

$$\prod_{p_i \leq x}\left(1 - \frac{1}{p_i}\right) \sim \frac{e^{-\gamma}}{logx} \tag{2.8}$$

Lemma2.5

$$\sum_{\substack{q=\prod p_i \\ p_i \leq \sqrt{N}}} \mu(q) = \sum_{i=0}^{\pi(\sqrt{N})} (-1)^i \binom{\pi(\sqrt{N})}{i} = 0 \tag{2.9}$$

Proof of Theorem2.3

We will prove an inequality with respect to $\pi^*(N)$ as follows.

$$\pi(\sqrt{N}) - 1 + N \cdot \left(\frac{C}{logN} - \varepsilon\right) < \pi^*(N) < \pi(\sqrt{N}) - 1 + N \cdot \left(\frac{C}{logN} + \varepsilon\right). \tag{2.10}$$

At first, we deform the third term of (2.3) as follows,

$$\sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q)\left(\frac{N}{q}-\frac{1}{2}\right) = \sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q)\cdot\frac{N}{q} - \frac{1}{2}\sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q) \tag{2.11}$$

Substitute (2.9) in (2.11), we get the equation as follows.

$$\sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q)\left(\frac{N}{q}-\frac{1}{2}\right) = \sum_{\substack{q=\prod p_i \\ p_i \le \sqrt{N}}} \mu(q)\cdot\frac{N}{q} = N\cdot\prod_{p_i \le \sqrt{N}}\left(1-\frac{1}{p_i}\right).$$

The third term of $\pi^*(N)$ in (2.3) can be written with N and Euler product.
On the other hand, for any positive number ε , there exists natural number N_0 satisfying that

if $N \ge N_0$, then $\left| \prod_{p_i \le \sqrt{N}}\left(1-\frac{1}{p_i}\right) - \frac{e^{-\gamma}}{log\sqrt{N}} \right| < \varepsilon$ by (2.8).

Thus we obtain an inequality as follows,

$$N\cdot\left(\frac{C}{logN}-\varepsilon\right) < N\cdot\prod_{p_i \le \sqrt{N}}\left(1-\frac{1}{p_i}\right) < N\cdot\left(\frac{C}{logN}+\varepsilon\right). \tag{2.12}$$

By adding $\pi(\sqrt{N})-1$ to all terms of (2.12), we obtain (2.10).
Furthermore we can obtain (2.7) as follows by (2.6) and (2.10).

$$\pi(\sqrt{N})-1+N\cdot\left(\frac{C}{logN}-\varepsilon\right)-2^{\pi(\sqrt{N})-1} < \pi(N) < \pi(\sqrt{N})-1+N\cdot\left(\frac{C}{logN}+\varepsilon\right)+2^{\pi(\sqrt{N})-1}.$$

□

We consider deeply about Euler product $\prod_{p_i \leq \sqrt{N}} \left(1 - \frac{1}{p_i}\right)$ in (2.12). Now let x be a real number and let p be a maximum prime less than x. S. Ramanujan gave the formula as follows.

$$\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{p}\right)} = e^{\gamma}\left(log\vartheta(x) + \frac{2}{\sqrt{x}} + S_1(x) + \frac{O(1)}{\sqrt{x}logx}\right) \quad (2.13)$$

Here $\vartheta(x)$ is Chebychev function defined by $\vartheta(x) = \sum_{p \leq x} logp, S_1(x) = -\sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho-1)}$.

ρ is non-trivial zeros of Riemann zeta function.

We take reciprocal the both side of (2.13) and as $x \to \infty$, we get as follows

$$\prod_{p_i \leq x} \left(1 - \frac{1}{p_i}\right) \sim \frac{e^{-\gamma}}{log\vartheta(x)}.$$

Thus we get as follows by (2.8).

$$\frac{e^{-\gamma}}{logx} \sim \frac{e^{-\gamma}}{log\vartheta(x)}$$

Thus $x \sim \vartheta(x)$. We know $logx \sim \frac{\vartheta(x)}{\pi(x)}$. Therefore we obtain as follows

$$\pi(x) \sim \frac{x}{logx}.$$

3. Legendre's conjecture of prime numbers

In this chapter we will consider Legendre's conjecture. Legendre's conjecture says that a prime number p satisfying $n^2 < p < (n+1)^2$ exists for any natural number n. Actually for arbitary n, we have.

$1^2 < 2, 3 < 2^2,$

$2^2 < 5, 7 < 3^2,$

$\cdots\cdots$

$4^2 < 17, 19, 23 < 5^2,$

$\cdots\cdots$

$6^2 < 37, 41, 43, 47 < 7^2,$

$\cdots\cdots$

$\cdots\cdots$

$10^2 < 101, 103, 107, 109, 113 < 11^2.$

Legendre's conjecture is not obvious, but existence of a prime number p satisfying $n^2 < p < (n+1)^2$ and consistence of $\pi((n+1)^2) - \pi(n^2) > 0$ is equivalent. So we guess Legendre expected by applying his formula to this problem.

Now we will consider the difference between $\pi(n^2)$ and $\pi((n+1)^2)$.

$$\pi(n^2) = \pi(n) - 1 + \sum_{\substack{q=\prod p_i \\ p_i \le n}} \mu(q) \left[\frac{n^2}{q}\right]$$

$$\pi((n+1)^2) = \pi(n+1) - 1 + \sum_{\substack{q=\prod p_i \\ p_i \le n+1}} \mu(q) \left[\frac{(n+1)^2}{q}\right]$$

$\pi(x)$ is monotonic increasing function, and third(principal) term of formula involving Möbius function $\sum \mu(q) \left[\frac{x}{q}\right]$ is monotonic increasing function too.

If $n+1$ is a prime number $(= p_k)$, next inequality consists.

$$\pi((n+1)^2) - \pi(n^2) > \pi(n+1) - \pi(n) = \mathrm{k} - (\mathrm{k}-1) = 1$$

therefore Legendre's conjecture is true.

Next we consider the case, that is, $n+1$ isn't a prime number. In that case, the difference $\pi((n+1)^2) - \pi(n^2)$ can be written as follows because of $\pi(n+1) = \pi(n)$.

$$\begin{aligned} \pi((n+1)^2) - \pi(n^2) &= \sum_{\substack{q=\prod p_i \\ p_i \le n}} \mu(q) \left[\frac{(n+1)^2}{q}\right] - \sum_{\substack{q=\prod p_i \\ p_i \le n}} \mu(q) \left[\frac{n^2}{q}\right] \\ &= \sum_{\substack{q=\prod p_i \\ p_i \le n}} \mu(q) \left\{\left[\frac{(n+1)^2}{q}\right] - \left[\frac{n^2}{q}\right]\right\} \end{aligned} \tag{3.1}$$

On the other hand quantity of $\left[\frac{(n+1)^2}{q}\right] - \left[\frac{n^2}{q}\right]$ is as follows,

$$\frac{2n+1}{q} - 1 < \left[\frac{(n+1)^2}{q}\right] - \left[\frac{n^2}{q}\right] < \frac{2n+1}{q} + 1$$

It shows that $\left[\frac{(n+1)^2}{q}\right] - \left[\frac{n^2}{q}\right]$ is equal to $\frac{2n+1}{q}$ approximately. We obtain approximate equation by replacing $\left[\frac{(n+1)^2}{q}\right] - \left[\frac{n^2}{q}\right]$ with $\frac{2n+1}{q}$.

$$\pi((n+1)^2) - \pi(n^2) \sim \sum_{\substack{q=\prod p_i \\ p_i \le n}} \mu(q) \frac{2n+1}{q} = (2n+1) \sum_{\substack{q=\prod p_i \\ p_i \le n}} \mu(q) \frac{1}{q}$$

$$\sim (2n+1) \frac{e^{-\gamma}}{logn} = C \frac{n}{logn} + o(1) > 0$$

where $C = 2e^{-\gamma}$.

This result gives authenticity of a fact Legendre's conjecture will consist.

4. New representation of $\pi(x)$ with Fourier series expansion and its estimation

First of all we prove a lemma as follows.

Lemma4.1

$$[x] = x - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi} \tag{4.1}$$

Proof

The difference staircase function $[x]$ and linear function $x - \frac{1}{2}$ is a periodic function $-x + n + \frac{1}{2}$ on half-closed interval $[n, n+1)$ for any integer more than 0. It is named sawtooth wave (Figure 1(B)).

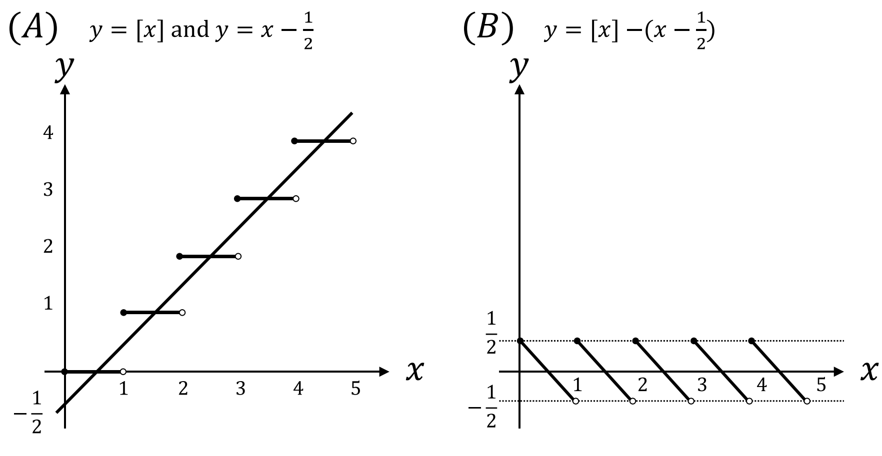

Figure1 (A) shows $y = [x]$ and $y = x - 1/2$. (B) shows $y = [x] - (x - \frac{1}{2})$.

When we usually deal with Fourier series, it's easy to set the period as 2π rather than 1. So now we consider the extension Euclidean space that's unit is 2π instead of 1 as Figure 2. Keeping in mind the staircase function f can be written as $f(x) = x - \pi + g(x)$,
Let's formulate the function g.

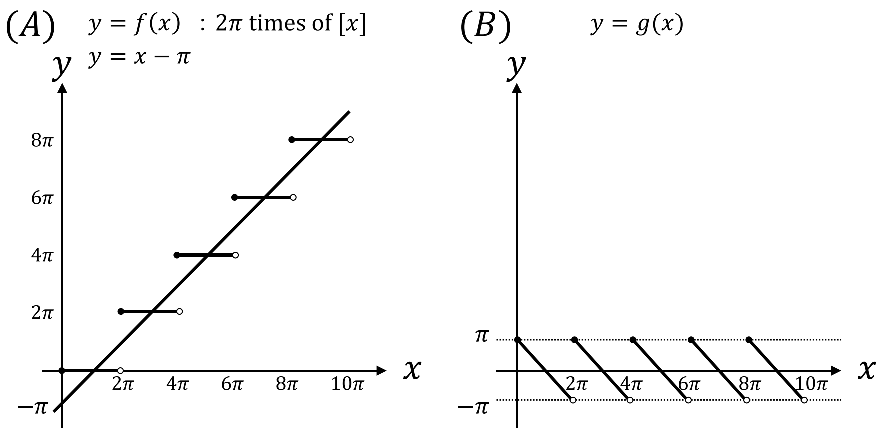

Figure2 (A) shows $y = f(x)$ that is 2π times of $y = [x]$, and $y = x - \pi$ that is 2π times of $y = x - 1/2$.
(B) shows $y = g(x) = f(x) - (x - \pi)$.

We will consider representation of $g(x)$ with Fourier series as follows,

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(x) \cdot \cos nx \, dx , \tag{4.2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} g(x) \cdot \sin nx \, dx .$$

We notice that $g(x)$ can be represented as follows on half-closed interval $[\, 0 \, , 2\pi)$.

$$g(x) = -x + \pi \tag{4.3}$$

We obtain representation of $g(x)$ from (4.2) and (4.3) with $a_n = 0$, $b_n = \frac{2}{n}$.

$$g(x) = \sum_{n=1}^{\infty} \frac{2 \sin nx}{n}$$

Therefore the function $f(x)$ is as follows

$$y = x - \pi + \sum_{n=1}^{\infty} \frac{2 \sin nx}{n} . \tag{4.4}$$

Next, we consider linear transformation $L_M : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by matrix M given below to obtain the formula of $[x]$.

$$M = \begin{pmatrix} \frac{1}{2\pi} & 0 \\ 0 & \frac{1}{2\pi} \end{pmatrix}$$

In this case, the image of function f by L_M is equal to function $[x]$. If we take any point $\begin{pmatrix} x \\ y \end{pmatrix}$ on the function f, then we set that the image of $\begin{pmatrix} x \\ y \end{pmatrix}$ by L_M is $\begin{pmatrix} x' \\ y' \end{pmatrix}$ on $[x]$. So, we get the equation $\begin{pmatrix} x' \\ y' \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}$.

By taking inverse, then we calculate that $\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2\pi & 0 \\ 0 & 2\pi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$

By substituting $x = 2\pi x'$ and $y = 2\pi y'$ in (4.4) ,

$$y' = x' - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin 2n\pi x'}{n\pi}$$

Thus we can obtain the equation of (4.1).

$\square$

Theorem4.2

If we define k_x as $\left[\frac{log\frac{log\,x}{log\,2}}{log\,2}\right]$ for any real number, then $\pi(x)$ is given as follows.

$$\pi(x) = \sum_{i=1}^{k_x-1} \sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{2^i}}}} \mu(q)\left(\frac{x^{\frac{1}{2^{i-1}}}}{q} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin 2n\pi \frac{x^{\frac{1}{2^{i-1}}}}{q}}{n\pi}\right) - k_x + 1 \tag{4.5}$$

Proof

We utilize Legendre's recurrence formula about $\pi(x)$, $\pi\left(x^{\frac{1}{2}}\right) \cdots \pi\left(x^{\frac{1}{2^{k-1}}}\right)$.

First equation is

$$\pi(x) = \pi\left(x^{\frac{1}{2}}\right) - 1 + \sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{2}}}} \mu(q)\left[\frac{x}{q}\right].$$

Second equation is

$$\pi\left(x^{\frac{1}{2}}\right) = \pi\left(x^{\frac{1}{4}}\right) - 1 + \sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{4}}}} \mu(q)\left[\frac{x^{\frac{1}{2}}}{q}\right].$$

Generally *i*-th equation is

$$\pi\left(x^{\frac{1}{2^{i-1}}}\right) = \pi\left(x^{\frac{1}{2^i}}\right) - 1 + \sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{2^i}}}} \mu(q)\left[\frac{x^{\frac{1}{2^{i-1}}}}{q}\right].$$

These equation is finite because $\pi\left(x^{\frac{1}{2^{i-1}}}\right)$ arrives 0 by descent. The number of equations is $\left\lfloor \frac{log\frac{log\,x}{log\,2}}{log\,2} \right\rfloor$. We simply write k_x instead of $\left\lfloor \frac{log\frac{log\,x}{log\,2}}{log\,2} \right\rfloor$. Especially k_x satisfies the condition of $x^{\frac{1}{2^{k_x+1}}} < 2 < x^{\frac{1}{2^{k_x}}}$ or $\pi\left(x^{\frac{1}{2^{k_x}}}\right) = 1$ and $\pi\left(x^{\frac{1}{2^{k_x+1}}}\right) = 0$.

We can get equality as follows, if we substitute respective equation to the first term above formula one by one.

$$\pi(x) = \sum_{i=1}^{k_x-1} \sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{2^i}}}} \mu(q) \left\lfloor \frac{x^{\frac{1}{2^{i-1}}}}{q} \right\rfloor - k_x + 1. \tag{4.6}$$

$\left\lfloor \frac{x^{\frac{1}{2^{i-1}}}}{q} \right\rfloor$ of (4.6) can be written by (4.1) as follows.

$$\left\lfloor \frac{x^{\frac{1}{2^{i-1}}}}{q} \right\rfloor = \frac{x^{\frac{1}{2^{i-1}}}}{q} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin 2n\pi \frac{x^{\frac{1}{2^{i-1}}}}{q}}{n\pi}. \tag{4.7}$$

By substituting (4.7) in (4.6), we can obtain (4.5).

□

Now we will expand the right hand of (4.5), that is

$$\pi(x) = \sum_{i=1}^{k_x-1} \sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{2^i}}}} \mu(q) \left(\frac{x^{\frac{1}{2^{i-1}}}}{q} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin 2n\pi \frac{x^{\frac{1}{2^{i-1}}}}{q}}{n\pi} \right) - k_x + 1.$$

$$\pi(x) = \sum_{i=1}^{k_x-1} \sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{2^i}}}} \mu(q) \frac{x^{\frac{1}{2^{i-1}}}}{q} + \sum_{i=1}^{k_x-1} \sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{2^i}}}} \mu(q) \sum_{n=1}^{\infty} \frac{\sin 2n\pi \frac{x^{\frac{1}{2^{i-1}}}}{q}}{n\pi} - k_x + 1. \tag{4.8}$$

We evaluate $\sum_{n=1}^{\infty} \frac{\sin 2n\pi \frac{x^{\frac{1}{2^{i-1}}}}{q}}{n\pi}$ to appear in the second term of (4.8).

$\sum_{n=1}^{\infty} \frac{\sin 2n\pi \frac{x^{\frac{1}{2^{i-1}}}}{q}}{n\pi}$ is equal to $\int_1^{\infty} \frac{\sin 2\pi \frac{x^{\frac{1}{2^{i-1}}}}{q} u}{\pi u} du$ approximately.

We put $\frac{x^{\frac{1}{2^{i-1}}}}{q} = X$, then integral $\int_1^{\infty} \frac{\sin 2\pi \frac{x^{\frac{1}{2^{i-1}}}}{q} u}{\pi u} du$ is

$$\int_1^{\infty} \frac{\sin 2\pi X u}{\pi u} du = \frac{1}{\pi}\left(\frac{\cos 2\pi X}{2\pi X} + \frac{\sin 2\pi X}{(2\pi X)^2} - \frac{2}{(2\pi X)^2} \int_1^{\infty} \frac{\sin 2\pi X u}{u^3} du\right) = O\left(\frac{1}{X}\right). \tag{4.9}$$

Here O is Landau's symbol. $\sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{2^i}}}} \mu(q) \sum_{n=1}^{\infty} \frac{\sin 2n\pi \frac{x^{\frac{1}{2^{i-1}}}}{q}}{n\pi}$ is series to exist positive and negative term in the range of $\left|\frac{1}{x}\right|$ except constant factor.

Thus we can expect $\sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{2^i}}}} \mu(q) \sum_{n=1}^{\infty} \frac{\sin 2n\pi \frac{x^{\frac{1}{2^{i-1}}}}{q}}{n\pi}$ is bounded.

Order of k_x is $loglogx$ for large x except constant factor.

Therefore the second term of (4.8) will be

$$\sum_{i=1}^{k_x-1} \sum_{\substack{q=\prod p_i \\ p_i \le x^{\frac{1}{2^i}}}} \mu(q) \sum_{n=1}^{\infty} \frac{\sin 2n\pi \frac{x^{\frac{1}{2^{i-1}}}}{q}}{n\pi} = O(loglogx)\,. \tag{4.10}$$

Under this condition, we can rewrite (4.8) as follows,

$$\pi(x) = C\,\frac{x}{logx} + o\left(x^{\frac{1}{2}}\right) + O(loglogx)\,. \tag{4.11}$$

The first term and second term of (4.11) is contribution from the term $\sum\sum \mu(q)\frac{x^{\frac{1}{2^k}}}{q}$ of (4.8), and the third term of (4.11) is contribution from (4.5) , (4.8) and (4.10).

Therefore we can expect as follows,

$$\pi(x) = C\,\frac{x}{logx} + o\left(x^{\frac{1}{2}} \cdot loglogx\right). \tag{4.12}$$

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